

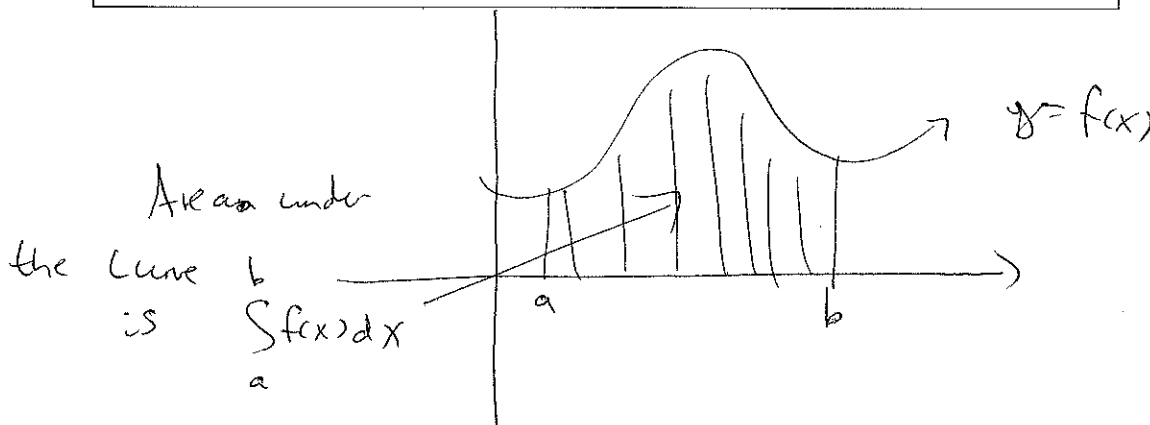
7.4 The Fundamental Theorem of Calculus

Area under the curve

Recall that if $f(x) \geq 0$,

$$\int_a^b f(x) dx$$

gives the area between the graph of $f(x)$ and the x -axis from $x = a$ to $x = b$.



If $f(x) = F'(x)$ gives the rate of change of $F(x)$, then the total change of $F(x)$ as x changes from a to b is given by the definite integral

$$\int_a^b f(x) dx.$$

We have that $F(x)$ is an antiderivative of $f(x)$.

We can write the total change in $F(x)$ from $x = a$ to $x = b$ as $F(b) - F(a)$.

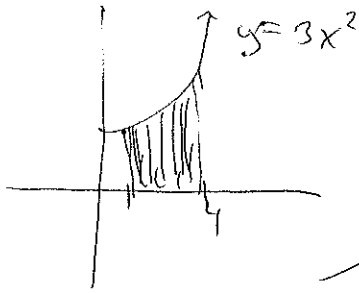
Fundamental Theorem of Calculus

Let f be continuous on the interval $[a, b]$, and let F be **any** antiderivative of f . Then

$$\int_a^b f(x) dx = F(b) - F(a) = \underline{F(x)} \Big|_a^b.$$

$$f(x) = 3x^2$$

Example. Use the Fundamental Theorem of Calculus to find $\int_1^4 3x^2 dx = F(4) - F(1)$



Definite integral

$$F(x) = \cancel{3x^3} + C = x^3 + C$$

$$\int_1^4 3x^2 dx = x^3 \Big|_1^4 = 4^3 - 1^3 = \underline{\underline{63}}$$

$F(4) - F(1)$

Notice that we have no constant C in the expression. The constant C is eliminated in the answer.

$$\text{Definite integral } \int_a^b f(x) dx = F(x) + C \Big|_a^b = F(b) + C - (F(a) + C)$$

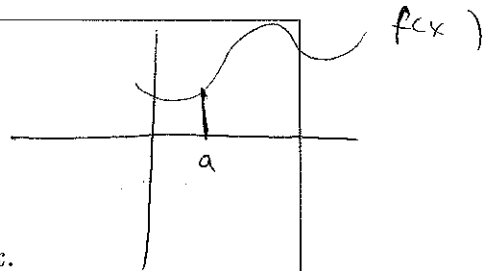
Indefinite integral

$$\int 3x^2 dx = x^3 + C$$

Properties of Definite Integrals

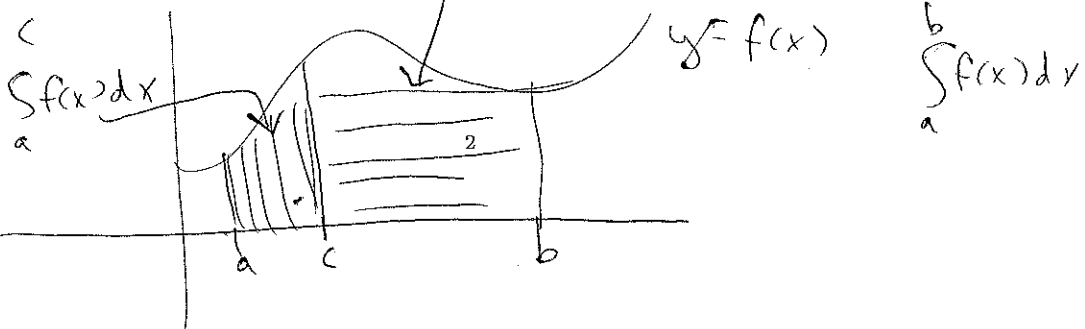
If all indicated definite integrals exist, we have

1. $\int_a^a f(x) dx = 0$
2. $\int_a^b k \cdot f(x) dx = k \int_a^b f(x) dx$ for any real constant k .
3. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$.
4. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.
5. $\int_a^b f(x) dx = - \int_b^a f(x) dx$.



$$\int_a^b f(x) dx$$

4.



$$f(x) = 3x^2 - x + 4$$

$$F(x) = x^3 - \frac{x^2}{2} + 4x$$

Example. Find $\int_2^7 (3x^2 - x + 4) dx$.

$$\begin{aligned} &= \left(x^3 - \frac{x^2}{2} + 4x \right) \Big|_2^7 \\ &= \left(7^3 - \frac{7^2}{2} + 4 \cdot 7 \right) - \left(2^3 - \frac{2^2}{2} + 4 \cdot 2 \right) \\ &= 346.5 - 14 = \underline{\underline{332.5}} \end{aligned}$$

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$

Example. Find $\int_0^4 x \sqrt{16-x^2} dx$.

1. method

$$u = 16 - x^2$$

$$du = -2x dx$$

$$\frac{du}{-2} = \frac{-2x dx}{-2}$$

$$-\frac{1}{2} du = x dx$$

$$\int x \sqrt{16-x^2} dx$$

$$= \int \sqrt{16-x^2} \cdot \underbrace{x dx}_{-\frac{1}{2} du} = \int \sqrt{u} \cdot \left(-\frac{1}{2}\right) du = -\frac{1}{2} \int \sqrt{u} du$$

$$= -\frac{1}{2} \int u^{\frac{1}{2}} du = -\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = -\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} = -\frac{1}{3} (16-x^2)^{\frac{3}{2}} = F(x)$$

$$\int_0^4 x \sqrt{16-x^2} dx = -\frac{1}{3} (16-x^2)^{\frac{3}{2}} \Big|_0^4 = -\frac{1}{3} \left[(16-4^2)^{\frac{3}{2}} - (16-0^2)^{\frac{3}{2}} \right]$$

$$= -\frac{1}{3} \cdot (-16)^{\frac{3}{2}}$$

$$= \frac{64}{3}$$

2. method:

$$u = 16 - x^2$$

when $x=0$,

$$\text{then } u = 16 - 0^2 = 16$$

when $x=4$,

$$\text{then } u = 16 - 4^2 = 0$$

$$\int_0^4 x \sqrt{16-x^2} dx$$

$$= -\frac{1}{2} \int_{16}^0 u^{\frac{1}{2}} du$$

Fundamental
thm of calculus

$$= \frac{1}{2} \int_0^{16} u^{\frac{1}{2}} du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_0^{16}$$

$$= \frac{1}{3} u^{\frac{3}{2}} \Big|_0^{16} = \frac{1}{3} 16^{\frac{3}{2}} - \frac{1}{3} 0^{\frac{3}{2}} = \frac{1}{3} 16^{\frac{3}{2}} = \frac{64}{3}$$

Property 5

$$= \frac{1}{2}$$

$$\int_0^{16} u^{\frac{1}{2}} du$$

Example. Find $\int_{-1}^2 \frac{x}{(x^2-5)^3} dx$.

$$u = x^2 - 5$$

$$du = 2x dx \quad \text{Divide by 2}$$

$$\frac{1}{2} du = x dx$$

When $x = -1$,
then $u = (-1)^2 - 5 = -4$

When $x = 2$,
then $u = 2^2 - 5 = -1$

2. method

$$\int_{-1}^2 \frac{1}{(x^2-5)^3} \underbrace{x dx}_{\frac{1}{2} du} = \int_{-4}^{-1} \frac{1}{u^3} \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int_{-4}^{-1} u^{-3} du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} \Big|_{-4}^{-1}$$

$$= \frac{1}{4} \Big|_{-4}^{-1} = -\frac{1}{4} \left[(-1)^{-2} - (-4)^{-2} \right] = \frac{1}{4} \left[1 - \frac{1}{16} \right]$$

$$= -\frac{1}{4} \left[1 - \frac{1}{16} \right]$$

$$= -\frac{1}{4} \left(\frac{15}{16} \right) = \underline{\underline{-\frac{15}{64}}}$$

$$= \frac{15}{4}$$

$$(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$$

Example. Suppose a company has the following rate of profit function (in thousand of dollars) after t years of operation,

$$P'(t) = (4t+4)(t^2+2t+3)^{1/3} dt.$$

marginal profit

(A) Find the total profit in the first two years.

Total profit in the first two years.

$$\int_0^2 P'(t) dt = \int_0^2 (4t+4)(t^2+2t+3)^{1/3} dt$$

$$u = t^2 + 2t + 3$$

$$du = (2t+2) dt$$

$$du = 2(t+1) dt$$

changing limits of integration!
when $t=0$,

$$u = 0^2 + 2 \cdot 0 + 3 = 3$$

when $t=2$,

$$u = 2^2 + 2 \cdot 2 + 3 = 11$$

$$= \int_0^2 4(t+1)(t^2+2t+3)^{1/3} dt$$

$$= \int_0^2 \underbrace{2}_{u^{1/3}} \cdot \underbrace{2(t+1)}_{du} (t^2+2t+3)^{1/3} dt$$

$$= 2 \int_0^2 (t^2+2t+3)^{1/3} \cdot 2(t+1) dt = 2 \int_3^{11} u^{1/3} du$$

$$= 2 \cdot \left[\frac{u^{4/3}}{4/3} \right]_3^{11} = 2 \cdot \frac{3}{4} \left[11^{4/3} - 3^{4/3} \right]$$

$$= \frac{3}{2} \left[11^{4/3} - 3^{4/3} \right]$$

$$= 30.2$$

Total profit in the first two years is

$$\underline{\underline{\$ 30,200}}$$

(B) Find the profit in the fifth year of operation.

$$\int_4^5 P'(t) dt = 2 \int_{27}^{38} u^{1/3} du$$

$$= \frac{3}{2} \left[u^{4/3} \right]_{27}^{38} = \frac{3}{2} \left[38^{4/3} - 27^{4/3} \right] = 70.13$$

Profit in the fifth year of operation is
\$ 70,130

when $t=4$,

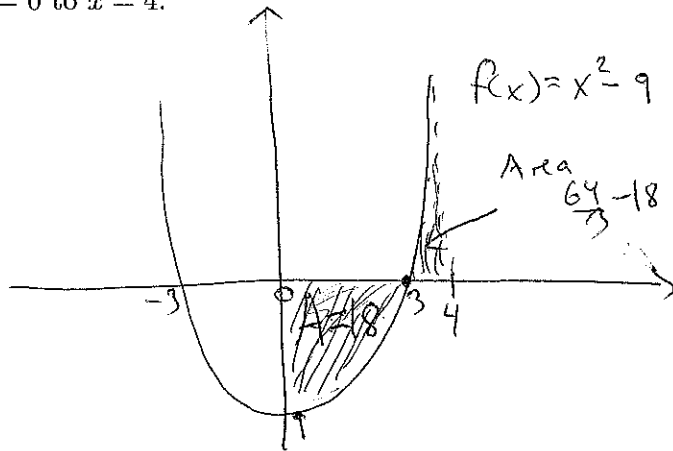
$$u = 4^2 + 2 \cdot 4 + 3 = 27$$

when $t=5$,

$$u = 5^2 + 2 \cdot 5 + 3 = 38$$

Area

Example. Find the area of the region between the x -axis and the graph of $f(x) = x^2 - 9$ from $x = 0$ to $x = 4$.



$$x^2 - 9 = 0$$

$$x = \pm 3$$

x -intercept is ± 3

If ~~f(x) is above the~~ $f(x) > 0$, then $\int f(x) dx$

If $f(x) > 0$ on $[a, b]$, then ~~Area~~ under the curve

$$\hookrightarrow A = \int_a^b f(x) dx.$$

If $f(x) < 0$ on $[a, b]$, then $A = - \int_a^b f(x) dx$

Area under the curve:

$$A = \left| \int_0^3 (x^2 - 9) dx \right| + \int_3^4 (x^2 - 9) dx$$

$$= \left| \left(\frac{x^3}{3} - 9x \right) \Big|_0^3 \right| + \left(\frac{x^3}{3} - 9x \right) \Big|_3^4$$

$$= \left| \frac{3^3}{3} - 9 \cdot 3 \right| + \left(\frac{4^3}{3} - 9 \cdot 4 - \left(\frac{3^3}{3} - 9 \cdot 3 \right) \right)$$

$$= |18| + \frac{64}{3} - 36 + 18 = 18 + \frac{64}{3} - 18 = \frac{64}{3}$$

Finding Area

To find the area bounded by $f(x)$, $x = a$, $x = b$, and the x -axis, use the following steps.

1. Sketch a graph.

2. Find any x -intercepts of $f(x)$ in $[a, b]$.

These divide the total region into subregions.

3. The definite integral will be **positive** for subregions above the x -axis and **negative** for subregions below the x -axis. Use separate integrals to find the areas of the subregions.

4. The total area is the sum of the areas of all subregions.